Backpaper Exam - Functional Analysis
M. Math I

07 June, 2022

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 50.

(iii) You may directly invoke results proved in the class.

(iv) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. (10 points) In this problem, we consider $\mathbb{C}^2$ as a Hilbert space with the standard inner product. Consider the linear map $T : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$T(\vec{x}, \vec{y}) = \Big(\frac{\vec{x}+\vec{y}}{2}, \frac{\vec{x}-\vec{y}}{2}\Big).$$

Compute the operator norm of T.

Total for Question 1: 10

2. For a given subspace M of a normed space $\mathfrak{X}$, answer (with justification) whether M is a codimension-one subspace of $\mathfrak{X}$.

   (a) (5 points) $\mathfrak{X} = \ell^1(\mathbb{N})$, and $M = \Big\{(x_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}) : \sum_{n=2}^{\infty} x_n = 2x_1\Big\}$

   (b) (5 points) $\mathfrak{X} = C[-1, 1]$, and $M = \Big\{f \in C[-1, 1] : \int_0^1 f(t)\, dt = \int_{-1}^1 f(t)\, dt.\Big\}$

Total for Question 2: 10

3. (10 points) Let $C^1[0,1]$ denote the space of C^1-functions on $[0,1]$ endowed with the sup norm inherited from $C[0,1]$. Let

$$\frac{d}{dx} : (C^1[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$$

be the differentiation operator. Prove that D is linear and has closed graph, but is not continuous.

Total for Question 3: 10

4. (10 points) A subset Y of a Hilbert space $\mathscr{H}$ is said to be an *orthonormal set* if it consists of mutually orthogonal unit vectors. If Y is an orthonormal set in $\mathscr{H}$, prove that the following conditions are equivalent:

   (i) $[Y] = \mathscr{H}$ (that is, the linear span of Y is dense in $\mathscr{H}$);

   (ii) For each $u \in \mathscr{H}$, $u = \sum_{y\in Y} \langle u, y\rangle y$;

Total for Question 4: 10

5. (10 points) Show that the closed unit ball of an infinite-dimensional Banach space is not compact (with respect to the norm topology).

Total for Question 5: 10